

A Problem-Based Approach to Mathematics Instruction

The way in which teachers approach mathematics instruction determines to a large extent the mathematics, thinking strategies, and dispositions that our students develop. Hiebert and others describe one of the essential principles for instruction focused on building understanding in mathematics: "Make the subject problematic." Instruction ought to allow "students to wonder why things are, to inquire, to search for solutions, and to resolve incongruities. It means that both curriculum and instruction should begin with problems, dilemmas, and questions for students" (1996, 12).

In a problem-based approach, students are expected to solve problems or make sense of mathematical situations for which no well-defined routines or procedures exist. In introductory activities, as well as in application settings, students are expected to explore problems, make conjectures, and draw generalizations about mathematics concepts and processes. Students can also make connections between mathematical ideas that are familiar to them by solving new problems in a variety of different settings. Although no one claims the existence of one correct way to teach, using good problems to plan instruction with the focus on student thinking and reasoning is one strategy that holds promise. Problem-based instruction in its simplest form is summarized by Gail Burrill in "Let's Talk about Mathematical Thinking and Reasoning" (1998), "Good teachers foster an environment in which the students do the work!"

A problem-based instructional approach is supported by recent research on teaching and learning in mathematics classes and has been strongly recommended by the National Council of Teachers of Mathematics in the *Professional Standards for Teaching Mathematics* (NCTM 1991). This article describes the preparation for instruction using a problem-based approach as part of the teaching-strategy repertoire, describes classes that use the approach successfully, and identifies pitfalls. The article suggests ways that mathematics teachers can get assistance in successfully implementing a problem-based-teaching approach. Finally, the arti-

cle includes research results discussing what students are likely to accomplish in classes where mathematics teachers use problem-based instruction successfully.

PREPARATION FOR INSTRUCTION

Choosing problems

Problem-solving tasks present situations in which no readily known or accessible procedure or algorithm determines the method of solution. The problem may come from a real-world application or a puzzling dilemma. The problem or task should be an activity that focuses students' attention on a particular mathematics concept, generalization, process, or way of thinking that matches the goals of school mathematics (NCTM 1989). The most important reason for choosing a problem or task is that the problem can "engage all of the students in the class in making and testing mathematical hypotheses" (Lampert 1990, 39). Example 1 shows such a problem, an introductory exploration to geometric series.

Example 1. Draw a square 1 unit on a side. Divide the square region into quarters, and shade the upper-left quarter. In the lower-right quarter of the original square region, do the same. Continue until you have a diagram that looks like **figure 1**. Find or estimate the sum of the areas of the shaded squares in as many ways as you can.

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Students solve problems for which no well-defined routines exist

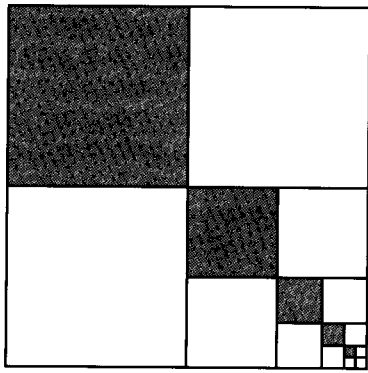


Fig. 1
The series problem:
finding the sum of the shaded areas

Example 1 offers opportunities for using multiple solution strategies, multiple representations of the concept, and mathematical communication that includes explanation and justification. Research suggests that these elements are important in a task chosen for this teaching approach (Brenner et al. 1997; de Lange 1993b; Stein, Grover, and Henningsen 1996). **Figures 2, 3, and 4** illustrate possible solution strategies and representations that students might use to grapple with this problem.

Henningsen and Stein (1997) summarize desirable features of the tasks using a problem-solving approach:

- Genuine problems that reflect the goals of school mathematics
- Motivating situations that consider students' interests and experiences, local contexts, puzzles, and applications
- Interesting tasks that have multiple solution strategies, multiple representations, and multiple solutions
- Rich opportunities for mathematical communication
- Appropriate content considering students' ability levels and prior knowledge
- Reasonable difficulty levels that challenge yet not discourage

Therefore, teachers must use their knowledge of mathematics and student thinking in choosing problems or tasks that "link with students' experience and for which students can see the relevance of the ideas and skills they already possess" (Hiebert et al. 1996, 16).

Example 2 is a problem that exhibits many of these features.

Example 2. To what extent are today's seventeen-year-olds like their ancestors?

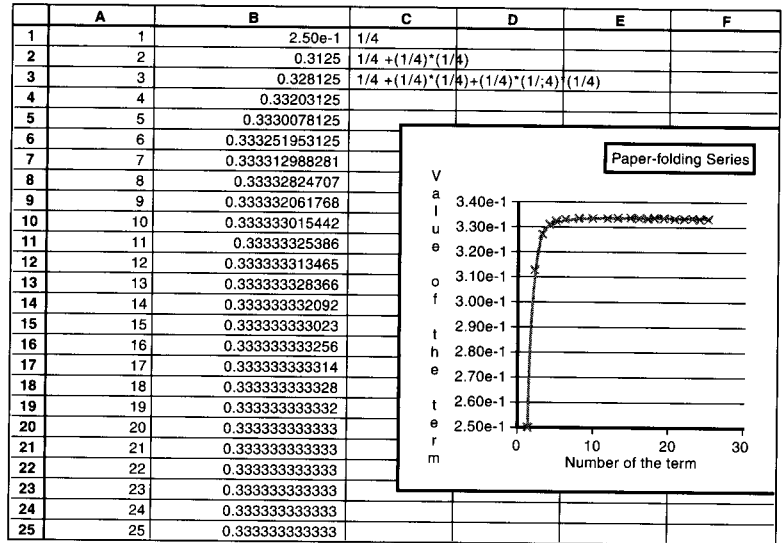


Fig. 2
Exploring the problem: Using a sequence of partial sums and graphing



Fig. 3
Exploring the problem: Using a geometric cut-and-paste technique to estimate the sum

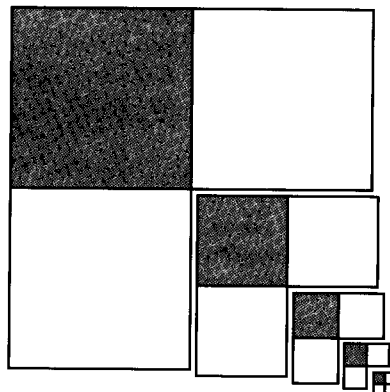


Fig. 4
Exploring the problem:
Using geometric reorganization

The problem offers opportunities for using multiple solution strategies

**Problems
may take
from a few
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to solve**

Small groups of students could tackle this problem in a unit on statistics that includes averages, measures of dispersion, lines of best fit, and chi-square tests or *t*-tests. Students could generate more specific questions, then collect data and use statistical methods to produce possible models and reasonable conjectures or answers to their posed questions. They could present their findings in writing or in presentations that reflect on the mathematics that they have learned. The questions that students might generate include comparison of such physical characteristics as height of a student and parents or grandparents, hobbies or school subjects in which parents or grandparents engaged at the same age, the number of children in succeeding generations of the same family, or the relationship between years of schooling and income or job satisfaction in different generations of the family.

Problem-solving tasks can be posed at a variety of places within classroom instructional sequences: to introduce a new mathematical idea, to develop understanding of an idea, or to apply the idea in a new situation. The problems may take from a few

minutes to a few weeks to explore and solve. Some problems can introduce or launch a new concept and take less than one class period, as in example 1. Others encourage decision making about relevant mathematics techniques and application of concepts and take several days, as in example 2.

At times, problems serve as the center for entire units of study. For example, in the Interactive Mathematics Program (Fendel et al. 1998), a multi-week unit called "The Overland Trail" sets a context for studying linear functions. Students set out to plan what to take on the trip across the western United States, estimate the cost of the move, and study rates of consumption and of travel. In a second example of substantial longer problem use, Boaler (1998) describes a high school mathematics department in which small groups of students are given projects that take two to three weeks to complete and are expected to be responsible for much of their own mathematics learning.

Planning principles

Time. In planning lessons, teachers must allow appropriate amounts of time for students to work on problem-solving tasks. If too little time is allowed, students often attempt to apply a known routine without connection to concepts, meaning, and understanding. If too much time is spent on a single activity when students are not making progress in constructing and understanding key mathematical ideas, students' work degenerates into unsystematic exploration or declines into no mathematical activity at all (Henningesen and Stein 1997). The length of time for the activity should be proportional to the size of the problem. Because students will probably not finish at the same time, the teacher should supply such alternative or extension activities as problems of the week to fill any time differential.

Organization. Problem-based lessons require instructional flexibility that responds to students' interaction with the problem. However, some points can be carefully and clearly planned ahead of time (Fendel et al. 1998; NCES 1996). **Figure 5** displays one possible lesson-plan outline for problem-based lessons. Note the essential elements: students can explore and make conjectures, listen to one another's explanations and justifications, and reach consensus about the concept or about how particular ideas are related.

Implementation of problem-based instruction. In this teaching approach, students make reasoned conjectures about a problem-solving task, justify their thinking, and listen to and consider other students' ideas. To make this type of classroom situation possible, the environment must be safe for stu-

- Objectives: Given the task, the student will be able to use reasoning and problem-solving techniques to explore solution strategies for the task, demonstrate conceptual understanding of the mathematics involved in the task, and communicate effectively both solution paths and solutions to the task.
- Possible sequence of events
 1. Plan the task, and decide on the degree of guidance to provide for students—from closely guided discovery to open-ended inquiry.
 2. Present the task in the large group; model the level of thinking required and the expectations for the written work and the final product.
 3. Organize work on the task by individuals, cooperative groups, or whole class.
 4. Assist individuals or groups by using planned key-question sequences, useful hints, or counterexamples rather than solution strategies.
 5. Ask students to share with the whole group their thinking about the processes used for solving the task, as well as solutions to the problem.
 6. Solicit other solution strategies from students in the class. Ask students to justify their reasoning for eliminating or accepting stated possibilities.
 7. Establish discovered results by class consensus, and summarize the major points of the lesson by asking pertinent questions of students. Write on the chalkboard.
 8. Give students a chance to practice using new knowledge in seatwork while checking for understanding.
 9. Plan students' application of new knowledge in novel situations that may launch future mathematics investigations.
- Assign homework. Allow students to practice procedures, apply concepts to new situations, invent new concepts or procedures, analyze other students' solutions, or generate proofs.
- Assess level of students' understanding through homework, open-ended performance assessments, projects, and portfolios by using holistic scoring guides.

Fig. 5

Outline of a lesson plan for a problem-based task

dents to express their thinking. Teachers must implement a code of behavior that encourages students to take risks by offering their ideas, conjectures, and conclusions without fear of reprisal, ridicule, or embarrassment. Students can be taught to use respectful language for questioning other students' conjectures, for example, "I want to question so and so's hypothesis" (Lampert 1990).

Students must also learn to value nonexamples and mistakes that can illuminate the problem situation or bring to light possible misconceptions about the mathematics. Mistakes are often as instructive as correct solution strategies in helping students understand the mathematics involved. Therefore, declaring mathematics classes to be no-pun zones, where students listen respectfully to one another and can change their conjectures during the discussion process, is helpful in implementing this approach. Students, however, must be held accountable for thoughtful, reasoned responses; careless or flippant comments or guesses from students distract many students and should be discouraged.

Additionally, teachers might establish and practice classroom routines so that students know what actions are expected (Clark 1997; Simon 1995). Teachers clarify expectations for—

- large-group discussion—taking turns, raising hands, explaining thinking, listening to others, paraphrasing other students' insights, respectfully differing with other people's ideas, clarifying one's own or others' thinking, justifying reasoning, revising conjectures;
- small-group activities—participating in specific roles, taking turns, brainstorming ideas, writing lists, conjecturing, developing solution strategies, asking questions, reporting back to the class;
- transitions from one part of class to the other—watching for a signal, listening to directions, heeding the time, handing in work to a designated place;
- behavior—following directions, moving about the room only as directed, taking time-outs, accepting consequences when discipline is needed.

Above all, successful teachers establish an atmosphere of mutual respect in which conjecture and justification are expected (Boaler 1998; Clark 1997; de Lange et al. 1993b).

Teachers face two challenges in a problem-based lesson, keeping the character of the problem-solving tasks from changing after the students begin working and keeping the cognitive demands of high-level tasks from declining (Stein, Grover, and Henningsen 1997). Participation in high-level reasoning and problem solving places uncomfortable demands on

some students and sometimes involves a perceived high personal risk for failure. Students will often try to get teachers to reduce the difficulty of the work. To use a problem-based approach successfully, the teacher must let students struggle together toward solutions without suggesting procedures (Clark 1997) yet provide sufficient scaffolding or guidance to keep students interested and on-task. Scaffolding may involve additional mathematics instruction for unknown content when necessary (Boaler 1998), individual handheld strategy prompt cards (Mevarech and Kramarski 1997), and hint cards (NCES 1996). Additionally, teacher questioning strategies to help students draw out their own thinking are important for success in this approach (Fendel et al. 1998).

During the public-reporting phase of the lesson, the teacher initiates and supports social interaction appropriate for making mathematics arguments in response to student conjectures (Lampert 1990). Teachers press students to give meaningful explanations (Henningsen and Stein 1997), demand accountability both for completing the task and learning the concepts and processes embedded in the task, and expect students to answer questions about mathematical assumptions and generalizations and justify their reasoning and conclusions.

ATTAINING TEACHING SUCCESS

Mathematics teachers who have successfully integrated problem-based-teaching approaches into their teaching repertoire report several factors that helped them implement the approach. One factor is finding problems that are engaging and at a suitable level of difficulty for a variety of students. Luckily, in addition to mathematics teaching journals such as this one, new curricula; such recent publications as Becker and Shimada (1997), Schoenfeld et al. (1999), NCTM's High School Addenda Series edited by Christian Hirsch, and de Lange et al. (1993a); and various Web sites allow access to many more problem-solving tasks than have been available to teachers in the past. For further information on new curricula, visit the NSF-funded K-12 Mathematics Curriculum Center Web site at www.edc.org/mcc/. For examples of additional Web sites, see forum.swarthmore.edu/students, which includes problems of the week and month, as well as many links to other problem collections.

Teachers can also develop more open-ended problems from existing problems in their own collection or textbooks by changing the wording of the problems. For example, suggested procedures or methods of solution could be deleted from the statement of the problem. Also, teachers can ask students to solve the problem in an alternative way, explain their thinking either orally or in writing, or propose an extension or generalization.

Successful teachers establish an atmosphere of mutual respect



Teaching expertise and a personal commitment to use these methods are two more crucial elements in developing a broadened instructional repertoire. However, since teachers tend to switch from new teaching methods for novel activities to more comfortable teaching methods, they may not be able to envision a successful implementation of the strategy without assistance from others. Some mathematics teachers have found mentors and coaches who help implement a new approach (Middle Grades Mathematics Project 1988). Teachers can consult a successful colleague down the hall or connect with an e-mail correspondent, join a grant-funded research or staff-development effort, or attend a summer course on problem-based instruction. Practical classroom assistance and emotional support are both helpful.

Another method that has been useful for teachers is cultivating a school or mathematics-department commitment to broadened teaching approaches. Japanese mathematics departments, for example, have a commitment to develop lessons and lesson sequences that successfully engage students in mathematical thinking and promote conceptual understanding (Stigler and Hiebert 1997). To implement this commitment, regular times each week are devoted to formulating new lessons and reconceptualizing other lessons or lesson sequences. The practice is viewed as a regular part of the department's responsibility.

STUDENT ACCOMPLISHMENT

The expansion of a teaching repertoire to include problem-based instructional strategies is a challenging task, but the effort has already been worthwhile for some teachers and mathematics programs. Research indicates that problem-solving, decision-making, modeling, and reasoning-process skills increase for students who learn from teachers using this approach. The evidence from high school studies (Boaler 1998; Webb 1997) supports this approach for developing these mathematics process skills. Also, enhanced conceptual understanding of such specific mathematics content as multiple representations for learning functions (Brenner et al. 1997; Kieran 1993; O'Callaghan 1998) and increased van Hiele levels in solving geometry problems (Swafford, Jones, and Thornton 1997) have been documented in classrooms where problem-based instruction is used as one teaching approach.

Moreover, new four-year high school curricula that employ problem-based strategies in the mathematics classroom have been created, and the results for students are promising. When students in classes using these curricula are compared on open-ended problem-solving tests with students in traditional classes, students using the new curricula do much better. When students in classes using

these curricula are compared on achievement tests with students in traditional classes, they usually do equally well. For example, the Systemic Initiative for Montana Mathematics and Science Project (SIMMS), using the PSAT for comparison, found that the "reform curriculum" demonstrated no differences in student performance on more traditional tests (Lott and Burke 1997). Students also enrolled in more mathematics and science courses after the new curriculum was adopted.

Moreover, diverse populations of students have been successful in classes using the problem-based approach. Research indicates that young women, English-as-a-second-language students, and students at a variety of different achievement levels attain higher results on average than in traditional mathematics classrooms (Boaler 1998; Brenner et al. 1997; Mevarech and Kramarski 1997). Fendel et al. (1998), studying IMP classrooms, and Lott and Burke (1997), studying SIMMS participants, found that instruction that emphasized reasoning, problem solving, and understanding helped high school students from poor communities with large, mixed ethnic and racial contingents improve their mathematical performance. Similar results have also been reported across a variety of cultures worldwide (Boaler 1998; Grunnetti and Jaquet 1996; Mevarech and Kramarski 1997; NCES 1996).

Problem-solving skills, conceptual understanding of specific mathematics content, and the ability to apply mathematics to difficult problems are worthy outcomes from mathematics study for high school students. If you want your students to think differently about mathematics, feel confident about tackling interesting and difficult problems, and take more mathematics courses, try implementing problem-based strategies in your classroom—and watch your students reap the benefits.

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