# Divisibility Rules for 3, 4, 6, and 9

- How do we know if a number is divisible by 2? (The last digit is even.)
- How do we know if a number is divisible by 10? (The last digit is 0.)
- How do we know if a number is divisible by 5? (The last digit is 0 or 5.)

These are the rules most people know, though most people do not know why they work. What about other numbers? How do we know if a number is divisible by 3? Write down the first ten multiples of 3.

$$3 \quad 6 \quad 9 \quad 12 \quad 15 \quad 18 \quad 21 \quad 24 \quad 27 \quad 30$$

**(GROUP ACTIVITY — 5 min.)** Does anyone see a pattern? (The sum of the digits is a multiple of 3.) Which numbers in this list are divisible by 6? (The ones that are even.) Do these conclusions work for the following numbers?

$$393 \quad 504 \quad 5832$$

(393 is divisible by 3, 504 is divisible by 3 and 6, 5832 is divisible by 3 and 6.)

Some people might remember the *rule of nines*. Can anyone tell me what that is? (If a number is a multiple of 9, then the sum of the digits is either 9, or a multiple of 9 whose digits add up to 9, or a multiple. . . . )

**(GROUP ACTIVITY — 5 min.)** What about the number 4? For instance, 12 is a multiple of 4, is there some way of adding up the digits to check this? No? Try writing down the first ten multiples of 4.

$$4 \quad 8 \quad 12 \quad 16 \quad 20 \quad 24 \quad 28 \quad 32 \quad 36 \quad 40$$

It is not obvious to us yet. What about the following multiples of 4

$$100 \quad 124 \quad 236 \quad 2328$$

It looks like a number is divisible by 4 if the last two digits are divisible by 4. Does anyone have any ideas how we could prove this?

Notice that

$$2328 = 2300 + 28 = 23(100) + 28.$$

We know that 100 is divisible by 4, so $2300 = 23(100)$ must be divisible by 4, so all we have to look at is whether the number formed by the last two digits (28) is divisible by 4.

We can use modular arithmetic to prove all these divisible rules. For instance, we can write

$$\begin{aligned} 2328 &\equiv 2(1000) + 3(100) + 2(10) + 8 \ mod\ 4 \\ &\equiv 2(10^3) + 3(10^2) + 2(10) + 8 \ mod\ 4 \end{aligned}$$

And since $100 \equiv 0 \ mod\ 4$, that means that $1000 \equiv 0 \ mod\ 4$, and $10^4 \equiv 0 \ mod\ 4$, and so on! So any digit past the first two we do not have to be concerned with when determining divisibility by 4.

$$\begin{aligned} 2328 &\equiv 2(0) + 3(0) + 2(10) + 8 \ mod\ 4 \\ &\equiv 28 \ mod\ 4 \end{aligned}$$

So modulo 4, any number is equivalent to the number formed by the last two digits. Of course, we cannot just leave $28 \ mod\ 4$, we have to keep going. Notice that $10 \equiv 2 \ mod\ 4$ (since '10 divided by 4' leaves a remainder of 2).

$$\begin{aligned} 2328 &\equiv 28 \ mod\ 4 \\ &\equiv 2(10) + 8 \ mod\ 4 \\ &\equiv 2(2) + 8 \ mod\ 4 \\ &\equiv 2(2) + 0 \ mod\ 4 \\ &\equiv 4 \ mod\ 4 \\ &\equiv 0 \ mod\ 4 \end{aligned}$$

This means that 2328 is divisible by 4.

**(INDIVIDUAL ACTIVITY — 5–10 min.)** Try to do something like this to prove the *rule of nines*. Use, for example, the number 97668.

$$\begin{aligned} 97668 &\equiv 9(10^4) + 7(10^3) + 6(10^2) + 6(10) + 8 \ mod\ 9 \\ &\equiv 9(1) + 7(1) + 6(1) + 6(1) + 8 \ mod\ 9 \\ &\equiv 9 + 7 + 6 + 6 + 8 \ mod\ 9 \\ &\equiv 36 \ mod\ 9 \\ &\equiv 3(10) + 6 \ mod\ 9 \\ &\equiv 3(1) + 6 \ mod\ 9 \\ &\equiv 9 \ mod\ 9 \\ &\equiv 0 \ mod\ 9 \end{aligned}$$

Since $10 \equiv 1 \ mod\ 9$, any number *mod* 9 is just the sum of its digits. For example,

$$28 \equiv 2 + 8 \equiv 10 \equiv 1 \ mod\ 9.$$

And since we know that $\frac{28}{9}$ leaves a remainder of 1 (remember that 27 is a multiple of 9), we know that this answer must be correct. And if a number is a multiple of 9, then it will be equivalent to 9 $mod$ 9 and thus equivalent to 0 $mod$ 9. For example,

$$\begin{aligned} 4626 &\equiv 4+6+2+6 \ mod \ 9 \\ &\equiv 18 \ mod \ 9 \\ &\equiv 1+8 \ mod \ 9 \\ &\equiv 9 \ mod \ 9 \\ &\equiv 0 \ mod \ 9 \end{aligned}$$

**(GROUP ACTIVITY — 10 min.)** Using what you have learned about modular arithmetic so far, try to discover the rule for divisibility by 8. You can try to do this with an example (like 4536), or using a general form for a number with $n$ digits, as in

$$\begin{aligned} a &= a_n a_{n-1} \cdots a_2 a_1 a_0 \\ &= a_n(10^n) + a_{n-1}(10^{n-1}) + \cdots + a_2(10^2) + a_1(10) + a_0 \end{aligned}$$

For example, in this notation, the $n = 4$ digit number 4536 would be written as

$$4(10^3) + 5(10^2) + 3(10) + 6$$

so $a_3 = 4$, $a_2 = 5$, $a_1 = 3$, and $a_0 = 6$.

So notice that $10 \equiv 2 \ mod \ 8$, $10^2 \equiv 4 \ mod \ 8$, $10^3 \equiv 0 \ mod \ 8$, ..., $10^n \equiv 0 \ mod \ 8$, so we have

$$\begin{aligned} a &\equiv a_n(10^n) + a_{n-1}(10^{n-1}) + \cdots + a_2(10^2) + a_1(10) + a_0 \ mod \ 8 \\ &\equiv a_n(0) + a_{n-1}(0) + \cdots + a_2(4) + a_1(2) + a_0 \ mod \ 8 \\ &\equiv a_2(4) + a_1(2) + a_0 \ mod \ 8 \end{aligned}$$

So a number $a$ is divisible by 8 if the number formed by the last three digits is divisible by 8, or alternatively, if the number formed by finding $4a_2 + 2a_1 + a_0$ is divisible by 8.